Journal of Engineering Mathematics **34:** 45–55, 1998.

On the sound radiated by a turbulent bubbly flow

L. VAN WIJNGAARDEN
J.M. Burgers Center for Fluid Dynamics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Received 4 July 1997; accepted in revised form 21 January 1998

Abstract. In this paper the sound emitted by a turbulent bubbly liquid is investigated, in particular for very dilute mixtures. The results are compared with those obtained by previous investigators, and with available experimental results.

Keywords: turbulence, bubbly flow, acoustics, Mach number, mixtures.

1. Introduction

The theory of sound emitted by a turbulent flow was initiated by Lighthill [1] in a celebrated paper in which he showed that, at distance x from the centre of a turbulent flow of density ρ, Mach number M and linear dimension L, with turbulence of r.m.s. fluctuation u and integral scale l, the emitted acoustic intensity I satisfies,

$$x^2 I \sim (\rho u U^2 L^2) M^5 \left(\frac{u}{U}\right)^5 \frac{L}{l}. \tag{1.1}$$

The Mach number is the mean velocity U divided by the sound velocity c of the undisturbed medium in which the observation point $\mathbf{x}$ is located.

Since the total available mechanical power in the turbulent flow is $\rho u U^2 L^2$, the fraction η of this, radiated as sound through a surface containing the observation point $\mathbf{x}$, is

$$\eta = M^5 \left(\frac{u}{U}\right)^5 \frac{L}{l}. \tag{1.2}$$

In the original paper [1] no explicit distinction was made between U and u, and then (1.1) shows Lighthill's famous result that the radiated sound and efficiency are proportional to U^8 and M^5. Since usually the Mach number of a turbulent flow is small, the efficiency of turbulence as a source of sound is small.

Crighton and Ffowcs Williams [2], henceforth denoted with CW, investigated how this is affected by the presence of small air bubbles in the turbulent region, in case the fluid is water. This is of great interest for underwater sound propagation, since in the upper layer of the oceans bubbles occur due to mixing with adjacent air, as a result of wave breaking. They found, by calculation, that the presence of bubbles increases the efficiency as a source of sound with a factor $(c/c_m)^4$ where c_m is the speed of sound in a bubbly liquid. Later, the same problem was studied theoretically by Prosperetti [3] and by Crighton *et. al.* [4], along slightly different lines. In Section 2 we shall briefly discuss these derivations and conclude that the mentioned increase of radiation efficiency holds good when the average distance between bubbles is small with respect to the integral scale l of the turbulence.

In the upper level of the oceans the gas concentration by volume, which we indicate with α, is often less that 10^{-4}. The gas concentration is also indicated as void fraction and is connected with the number density n and the average bubble radius a by

$$\alpha = \tfrac{4}{3}\pi n a^3. \tag{1.3}$$

With a bubble radius of 10^{-3} m and a void fraction of 10^{-4}, the average distance between bubbles is about 3 cm. In the intense turbulence in breaking waves this is often comparable with the integral scale. Prosperetti [3], for example, takes for this 0·1 m.

Under such conditions, regarding each bubble as if radiating into an effective medium with sound velocity c, is not an appropriate model. Another way of looking at the problem is to start with the exact expression for c_m,

$$c_m^{-2} = c^{-2}(1-\alpha)^2 + c_g^{-2}\alpha^2 + \alpha(1-\alpha)\rho/p. \tag{1.4}$$

Here c_g is the sound velocity in the gas phase and p is the pressure. For $\alpha \to 0$, this gives, if we assume $\rho c^2/p$ to be large with respect to unity:

$$c^2/c_m^2 = 1 + \alpha\rho c^2/p. \tag{1.5}$$

This shows that, in agreement with the general theory of nonhomogeneous fluids (see *e.g.* Batchelor [5]), the first correction on the speed of sound produced by the bubbles is of order α. In the theory of CW, it is assumed that $\alpha\rho c^2/p$ is large with respect to one. Then $c_m^2 = p/\alpha\rho$ and the factor η mentioned above, with which CW find the emitted sound intensity to be multiplied with respect to the single fluid case, is proportional to α^2. This indicates already interaction between the bubbles, exemplified in the effective medium model. In this paper we search for a contribution of order α, valid when bubbles are so far apart that their interaction is negligible. However, their distance is no longer small with respect to the integral scale of the turbulence.

In Section 3 we make this calculation and in Section 4 we discuss another contribution to the emitted sound of the same order, *viz.* that which is produced by the interaction between the pressure fluctuations of the unperturbed (by the presence of the bubbles) turbulence and velocity fluctuations caused by the volume oscillations of the bubbles.

2. Sound radiated from a region with a moderately large bubble concentration

We start with the original equation given in [1] for a single phase turbulent flow. The density fluctuation obeys

$$\frac{\partial^2\rho}{\partial t^2} - c^2\nabla^2\rho = \frac{\partial^2}{\partial x_i \partial x_j}\{\rho u_i u_j + [(p-p_0) - c^2(\rho-\rho_0)]\delta_{ij}\}. \tag{2.1}$$

Here δ_{ij} is the Kronecker delta and $-\rho u_i u_j$ the instantaneous Reynolds stresses, which are large with respect to the viscous stresses. Outside the turbulent region the Reynolds stresses are zero and the pressure perturbation $p - p_0$ equals $c^2(\rho - \rho_0)$, so that the right-hand side of (2.1) vanishes there. Crighton *et al*. [4] define

$$\rho_e = (\rho - \rho_0) - (p - p_0)/c^2 \tag{2.2}$$

and convert (2.1) into

$$c^{-2}\frac{\partial^2}{\partial t^2}(p-p_0)-\nabla^2(p-p_0)=\frac{\partial^2}{\partial x_i\partial x_j}(\rho u_i u_j)-\frac{\partial^2\rho_e}{\partial t^2}. \tag{2.3}$$

Next we consider bubbles to be introduced into the turbulent region, with concentration α by volume.

(i) CW.
The equation of mass conservation for the fluid becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=Q, \tag{2.4}$$

where Q is a source of mass of magnitude

$$Q=-\rho\frac{\mathrm{D}}{\mathrm{D}t}(1-\alpha)=\rho\frac{\mathrm{D}}{\mathrm{D}t}\alpha. \tag{2.5}$$

This leads to a 'Lighthill Equation' for the fluid with an extra term at the right-hand side:

$$\frac{\partial^2\rho}{\partial t^2}-c^2\nabla^2\rho=\frac{\partial Q}{\partial t}+\frac{\partial^2}{\partial x_i\partial x_j}\{\rho u_i u_j+[(p-p_0)-c^2(\rho-\rho_0)]\delta_{ij}\}. \tag{2.6}$$

The formulation (2.4)–(2.6) implies an averaging over a region containing many bubbles. This region must have a linear dimension small with respect to macroscopic lengths such as L and l, but large with respect to the inter-bubble distance $n^{-1/3}$. The averaging must be an ensemble averaging, the ensemble consisting of all possible configurations of a large number of bubbles in the volume on the mesoscale. As is well known (see *e.g.* [5]) the calculation of multiple interactions beyond pair interactions meets with great difficulties and therefore approximations have to be made. In the present case it is as if there were locally in the liquid a volume source produced by fluctuations in the volume of all the bubbles in the considered portion of the liquid together. In CW it is shown that in (2.6) the term with the rate of change of Q is the most important one. Under the assumption of isothermal behaviour this can be written as $-c_m^{-2}\frac{\partial}{\partial t}\frac{\mathrm{D}p}{\mathrm{D}t}$ and herewith (2.6) becomes:

$$\frac{\partial^2\rho}{\partial t^2}-c^2\nabla^2\rho=-c_m^{-2}\frac{\partial}{\partial t}\frac{\mathrm{D}p}{\mathrm{D}t}, \tag{2.7}$$

where the right-hand side is to be evaluated inside the two-phase region.

By analysis of the solution of this equation CW, show that the sound intensity at x is now given by

$$x^2 I\sim\frac{1}{4\pi}(\rho u U^2 L^2)\left(\frac{u}{U}\right)^5 M^5\left(\frac{c}{c_m}\right)^4\frac{L}{l} \tag{2.8}$$

Comparison with (1.1) shows that the intensity has increased with a factor of magnitude $(c/c_m)^4$, which, as they point out, can be as large as 10^7. They denote as efficiency the fraction of the total power radiated through a sphere of radius x, *i.e.* the quantity η given in (1.2) for the single-phase case. With bubbles, this efficiency is therefore

$$\eta = \left(\frac{u}{U}\right)^5 M^5 \left(\frac{c}{c_m}\right)^4 \frac{L}{l}. \tag{2.9}$$

(ii) Crighton *et al*. [4]. These authors write for $\partial\rho_e/\partial t$ inside the two phase region, using (2.2) and the equations of motion,

$$\frac{\partial\rho_e}{\partial t} = -\frac{l}{c^2}\left[\left\{1 - \frac{\rho c^2}{\rho_m c_m^2}\right\}\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{p - p_0}{\rho}\frac{\mathrm{D}\rho_m}{\mathrm{D}t}\right], \tag{2.10}$$

where ρ_m denotes the density in the two-phase flow. Substituting this in (2.3), they obtain subsequently for the pressure perturbation:

$$c^{-2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2}{\partial x_i \partial x_j}\rho u_i u_j + c^{-2}\frac{\partial}{\partial t}\left[\left\{1 - \frac{\rho c^2}{\rho_m c_m^2}\right\}\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{p - p_0}{\rho_m}\frac{\mathrm{D}\rho_m}{\mathrm{D}t}\right]. \tag{2.11}$$

Again, the term containing the factor c^2/c_m^2 is dominant and the resulting equation is within the accuracy of the approximation equivalent with (2.7).

This derivation clearly shows the bubbly region as a fictitious medium with sound velocity c_m in which there is a continuous distribution of monopoles.

(iii) Prosperetti [3]. This author follows basically the same route as in [4]. He starts by writing down the Lighthill equation (2.1) and observes directly that inside the bubbly turbulent region $p - p_0$ is not equal to $c^2(\rho - \rho_0)$ but rather to $(\mathrm{d}p/\mathrm{d}\rho)_s(\rho - \rho_0)$, which is $c_m^2(\rho - \rho_0)$. This leads after some manipulation to transform the double space derivative into a double time derivative again to an equation of the type (2.7). Incidentally, it is interesting that Prosperetti [3] stresses the monopole character of the sound produced by the bubbles (he deals with a free surface and then it becomes dipole sound), whereas CW, in discussing (2.8) write 'The dependence of I on M is rather surprising being characteristic of quadrupole rather than monopole sources. It is less surprising if we remember that it was noted that the whole problem could be tackled using a quadrupole type of source only. The monopole Q is equivalent, in part, to $\partial/\partial t(p - \rho c^2)$, a quadrupole time derivative which would occur in this alternative treatment, p and ρ now both referring to the two-phase mixture'. The alternative treatment, mentioned in this quotation, is in fact given in Prosperetti [3], who, however, considers it as monopole sound. The present author tends to side with Prosperetti here. Obviously, the dependence on the Mach number is M^{-4}, the dependence on c is, however, as c^{-1} due to the factor $(c/c_m)^4$. The dependence on c is typical for monopole sources. The monopole character will become even more clear in the next section where we consider the case of very small values of α, to be precise, of such small values that the inter bubble distance $n^{-1/3}$ is not necessarily small with respect to the integral scale l of the turbulence.

3. Sound radiated from a dilute turbulent bubbly flow

We consider a large volume of linear dimension L, occupied by a turbulent flow in which bubbles are dispersed. The bubbles are spherical, radius a, and randomly dispersed in the fluid. Let us assume that there are N bubbles in the volume. We want to know the sound

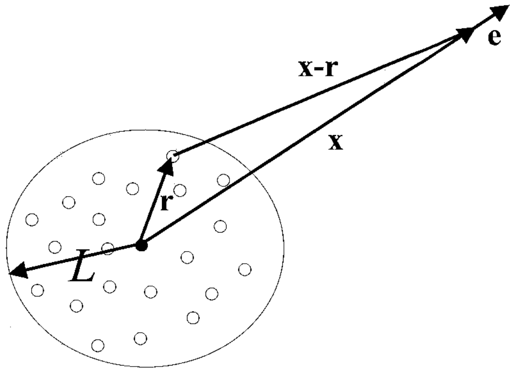

Figure 1. The sphere of radius L contains turbulent fluid and in addition air bubbles. Sound is radiated and measured in **x**.

intensity observed in a point **x** located outside the volume (see Figure 1), where the fluid is neither turbulent nor bubbly. As we have seen, in the absence of bubbles this leads to (1.1). We are interested in the additional intensity in **x** due to bubbles. We assume that these are so small that the turbulent pressure fluctuation $p' - p_0$ is uniform over the surface of a bubble. Each bubble performs volume oscillations as a result of the turbulence and radiates sound. Let in $\mathbf{x}(p - p_0)$ be the pressure perturbation and $\mathbf{v} \cdot \mathbf{e}$ the velocity in the direction **e**. Then the intensity is $(p - p_0)\mathbf{v} \cdot \mathbf{e}$. In the problem, there are two types of fluctuation. The first is due to the turbulence, and we shall indicate averaging over the turbulent time scale with an overbar, in the case of the intensity $\overline{(p - p_0)\mathbf{v} \cdot \mathbf{e}}$. Next, there are the fluctuations caused by the fluctuating positions of the bubbles.

A configuration of the N bubbles is indicated with C_N. The probability density of such a configuration is, say, $F(C_N)$. The average over all possible configurations is the ensemble average, which we indicate with $\langle\,\rangle$. Using these definitions we have,

$$I = \int \{\overline{(p - p_0)\mathbf{v} \cdot \mathbf{e}}\} F(C_N)\, \mathrm{d}C_N = \langle\, \overline{(p - p_0)\mathbf{v} \cdot \mathbf{e}}\, \rangle. \tag{3.1}$$

It is well known (see *e.g.* [5]) that in the lowest approximation in the volume concentration α each configuration consists of only one bubble. Let the position of the centre of this be **r**. Then the ensemble averaging means averaging over **r**. We start therefore with calculating $\overline{(p - p_0)\mathbf{v} \cdot \mathbf{e}}$ at **x** due to a bubble at **r**.

The oscillating bubble is a monopole source of strength m, say, with acoustic potential induced at **x**,

$$\varphi = -m\left(t - \frac{|\mathbf{x} - \mathbf{r}|}{c}\right) \Big/ 4\pi |\mathbf{x} - \mathbf{r}|. \tag{3.2}$$

The strength m is related to the turbulent pressure fluctuation in the following way. The compression and expansion of the bubble is, to a good approximation, adiabatic, [6], and we have, a_0 being the undisturbed radius of the bubble,

$$(a/a_0)^{-3\gamma} = p'/p_0 \tag{3.3}$$

where γ is the ratio of specific heats in the gas.

Linearization with respect to a_0 and p_0 gives

$$m = 4\pi a_0^2 \frac{\mathrm{d}a}{\mathrm{d}t} = -4\pi \left(\frac{a_0^3}{3\gamma p_0} \right) \frac{\mathrm{D}}{\mathrm{D}t}(p' - p_0). \tag{3.4}$$

We denote the quantity $a_0^3/3\gamma p_0$ by K, and the time derivative of $(p' - p_0)$ by $\dot{P}$,

$$\frac{a_0^3}{3\gamma p_0} = K \tag{3.5}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(p' - p_0) = \dot{P}. \tag{3.6}$$

Further we may neglect a with respect to x and r and we shall in the following make the abbreviation

$$s = |\mathbf{x} - \mathbf{r}|. \tag{3.7}$$

Introducing (3.4)–(3.7) into (3.2), we have

$$\varphi = \frac{K\dot{P}(t - s/c)}{s}. \tag{3.8}$$

Whereas without bubbles the pressure fluctuation P can express itself acoustically only as a quadrupole, it now generates a monopole. The velocity which this acoustic monopole induces at $\mathbf{x}$ is

$$\mathbf{v} = \frac{\partial\varphi}{\partial s}\frac{\mathbf{s}}{s} = -K \left\{ \frac{\dot{P}(t - s/c)}{s^3} + \frac{\ddot{P}(t - s/c)}{cs^2} \right\} \mathbf{s}. \tag{3.9}$$

Likewise, the pressure perturbation caused at $\mathbf{x}$ by the bubble at $\mathbf{r}$ is

$$p - p_0 = -\rho \frac{\partial\varphi}{\partial t} = -K\rho \frac{\ddot{P}(t - s/c)}{s}. \tag{3.10}$$

For one realisation the time average of $(p - p_0)\mathbf{v}$ is found from the product of the right-hand sides of (3.9) with that of (3.10). Subsequently, we take the ensemble average by multiplying with the probability of finding a bubble centre in a volume element $\mathrm{d}^3\mathbf{r}$ around $\mathbf{r}$ and integrating over all possible values of $\mathbf{r}$. We shall take here the most simple case and take the probability $F(\mathbf{r})\,\mathrm{d}^3\mathbf{r}$ of finding a bubble centre in $\mathrm{d}^3\mathbf{r}$ to be uniform and equal to $n\,\mathrm{d}^3\mathbf{r}$. Further, since we want to make an estimate of the sound emitted by the turbulent patch, as Prosperetti [3] calls these regions, we can take it most simply to be a sphere with radius L. Then, from (3.1), (3.8)–(3.10), we obtain

$$\begin{aligned} \langle\, \overline{(p - p_0)\mathbf{v}\cdot\mathbf{e}}\,\rangle &= 2\pi \int_0^L nr^2\,\mathrm{d}r \int_0^\pi \sin\theta\,\mathrm{d}\theta\,\overline{(p - p_0)\mathbf{v}\cdot\mathbf{e}} \\ &= 2\pi n\rho K^2 \int_0^L r^2\,\mathrm{d}r \int_0^\pi \sin\theta\,\mathrm{d}\theta \left\{ \frac{\ddot{P}(t - s/c)}{s} \right\} \\ &\qquad \times \left\{ \frac{\dot{P}(t - s/c)}{s^2} + \frac{\ddot{P}(t - s/c)}{cs} \right\} \frac{\mathbf{s}\cdot\mathbf{e}}{s}. \end{aligned} \tag{3.11}$$

The quantity $\ddot{P}\dot{P}$ in (3.11) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{P}^2)$$

and does not survive the time averaging. From Figure 1 it follows that

$$\frac{\mathbf{s}\cdot\mathbf{e}}{s} = \frac{(x - r\cos\theta)}{(x^2 - 2xr\cos\theta + r^2)^{1/2}}. \tag{3.12}$$

Inserting (3.12) into (3.11), we obtain after time averaging and integrating over θ and r,

$$x^2 I = \frac{4\pi n\rho\overline{\ddot{P}}^2 K^2 L^3}{3c}. \tag{3.13}$$

It is interesting to note that (3.13) is not a so-called far-field approximation, which would imply that $x \gg L$ and would involve, in working out (3.11) approximation based on that.

No such assumption has been made here. The only assumption regarding the geometry is that the bubbly blob has a spherical shape.

Using the definitions (1.3) and (3.5), we find that the result (3.13) becomes

$$x^2 I = \frac{4\alpha\overline{\ddot{P}}^2 a_0^3 L^3}{9\gamma^2 p_0^2 c}. \tag{3.14}$$

To give an estimate of the magnitude of the quantity at the right-hand side of (3.14) , we proceed as CW do and take the pressure fluctuation $P = p' - p_0$ equal to $\rho u U$. A measure for the time derivative applied to this is the turbulent frequency u/l. We make no distinction between Eulerian and Lagrangian frequencies. Inserting this into (3.14) and grouping quantities together, we have

$$x^2 I = \frac{1}{9}\alpha(\rho u U^2 L^2)\left(\frac{\rho U^2}{\gamma p_0}\right)^2\left(\frac{a_0}{l}\right)^3\left(\frac{L}{l}\right)\left(\frac{u}{U}\right)^5 M, \tag{3.15}$$

so the ratio of emitted sound intensity with bubbles to that without bubbles, which we call here the gain G, is, from (1.1) and (3.15),

$$G = 1 + \frac{1}{9}\alpha\left(\frac{\rho U^2}{\gamma p_0}\right)^2\left(\frac{a}{l}\right)^3 M^{-4}. \tag{3.16}$$

As discussed in the previous sections, the gain obtained by CW is $(c/c_m)^4$, which we can also write, using $c_m^2 = \gamma p/\alpha\rho$, preferable to $p/\rho\alpha$, since bubbles of radius of a few mm behave adiabatically rather than isothermally,

$$G = 1 + \alpha^2\left(\frac{\rho U^2}{\gamma p_0}\right) M^{-4}. \tag{3.17}$$

The difference between (3.16) and (3.17) is that in (3.16) the correction due to the presence of the bubbles is proportional to α and in (3.17) to α^2. Moreover, the (small) quantity $(a_0/l)^3$

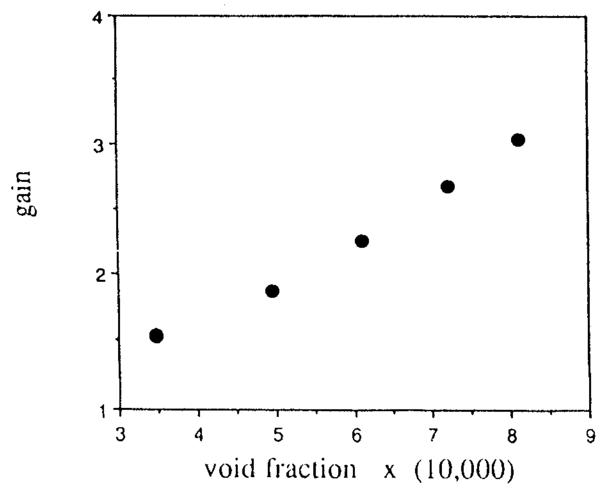

Figure 2. Gain, caused by bubbles, of sound radiation from a turbulent region as measured by Korman *et al*. [7].

appears in (3.17). Another difference is that (3.17) holds in the far field, whereas no such restriction is present for (3.16). It would be interesting to be able to compare these results with experiments. However, data for which all the values of quantities occurring in these relations are given are scarce. The present author has found some interesting results in Korman, Roy and Crum [7]. These authors made measurements in the near field of a turbulent jet in water, with and without bubbles. In their experiments the void fraction α is of the order of 10^{-4}. At the nozzle the velocity of the jet is 12 m/s. No values for a_0 or for l are given. At these relatively high velocities, reasonable values for a and l are 10^{-3} m and 10^{-2} m, respectively.

Using these values and in addition $c = 1500\,m/s$, $p_0 = 10^{-5}$ Pa and $\gamma = 1{\cdot}4$, we find for (3.16)

$$G = 1 + 2{\cdot}7 \times 10^4\alpha. \tag{3.18}$$

Likewise, (3.17) results with the same values in

$$G = 1 + 2{\cdot}4 \times 10^8\alpha^2.$$

In Figure 2, from [7], are shown their experimental results, for the gain G. It is clear that this increases with α rather than α^2. Apart from the first few points, the data are reasonably represented with

$$G = 1 + 0{\cdot}22 \times 10^4\alpha. \tag{3.19}$$

This is much smaller than predicted by our result (3.18). However, the magnitude of G is very sensitive for the value of a/l, *cf.* (3.16). Neither the value of a nor that of l is given in [7]. If we take l equal to 2·3 cm instead of 1 cm, there is complete agreement between the right-hand sides of (3.18) and (3.19) for the experiments reported in [7]. This emphasises the need for more experimental data.

The behaviour, proportional to α, which corresponds with the experimental results summarised in (3.19), makes it worthwhile to try to improve on some estimates on which the

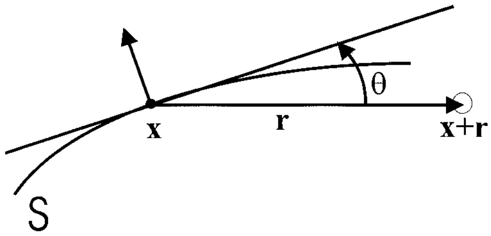

Figure 3. S is the boundary of the turbulent bubbly flow. **x** is on this surface and receives velocity fluctuations from a bubble in $\mathbf{x}+\mathbf{r}$, excited by the turbulent pressure fluctuation there.

right-hand side of (3.16) is based. For example, concerning the spherical shape of the turbulent region, the distinction between Eulerian and Lagrangian correlations and the relative magnitude of inter-bubble distance with respect to turbulent scales. That is left for future work.

At this stage it seems fair to conclude that there is, through the mechanism described here, an order-α contribution to the radiated noise. Of course, this does not change in any sense the validity of the results obtained by CW and others, of order α^2, and described in Section 2 of the present paper.

In this paper we want to conclude with the observation that, while in the work of CW the term with Q in (2.5) is dominant, this is not necessarily the case for the very dilute suspensions studied here. This forms the subject of the final section.

4. Additional contributions to the emitted sound

In evaluating $\langle\, \overline{(p-p_0)\mathbf{v}\cdot\mathbf{e}}\,\rangle$ (the right-hand side of 3.1) we have calculated the product of the pressure fluctuations caused by the bubbles with the velocity fluctuations caused by the same ones. However, a contribution is also made by the original turbulent pressure fluctuations $(p'-p_0)$, and denoted by us with P, multiplied by the velocity fluctuations produced by the bubbles, and vice versa. In the Lighthill type of equation (2.6), such contributions would appear in the terms $u_i u_j$ in the right-hand side, one u being the original velocity fluctuation and the other being induced by the bubbles. Such terms are difficult to estimate in the CW approach. In CW there was no need for that, since the term with Q overshadows all the others. For the very dilute case this is not so certain, as we shall now demonstrate.

We start with observing, referring to Figure 1, that in the absence of dissipation, all the energy flowing through a closed surface containing **x** also flows through the bounding surface S of the turbulent volume, in our case a sphere of radius L. Consider a unit surface area with centre **x** now situated on the surface of the turbulent region. Let **n** be the outward normal at **x** and let there be a bubble with centre $\mathbf{x}+\mathbf{r}$ (Figure 3). We calculate now

$$\langle\, \overline{(p-p_0)\mathbf{v}\cdot\mathbf{n}}\,\rangle$$

through a unit surface at **x**, where **v** is the velocity in **x** induced by a bubble in $\mathbf{x}+\mathbf{r}$. The latter is always given by the potential (3.8); K and P are defined in (3.5) and (3.6), respectively. Hence, using (3.9), we find for the intensity radiated in the direction **n** through a unit surface S around **x**

$$-nK\left[\int \frac{\overline{P(\mathbf{x},t)\dot{P}(\mathbf{x}+\mathbf{r},t-r/c)}\mathbf{n}\cdot\mathbf{r}\,\mathrm{d}^3\mathbf{r}}{r^3}+\int \frac{\overline{P(\mathbf{x},t)\ddot{P}(\mathbf{x}+\mathbf{r},t-r/c)}\mathbf{n}\cdot\mathbf{r}\,\mathrm{d}^3\mathbf{r}}{cr^2}\right] \qquad (4.1)$$

To evaluate the above integrals, we draw an r, θ, ψ spherical polar axes system with origin in $\mathbf{x}$, such that θ is the angle between $\mathbf{r}$ and the tangent to S and the turbulent region is in $0 < \theta < \pi, 0 < \psi < \pi, a < r < \infty$. With this choice the quantity $\mathbf{n} \cdot \mathbf{r}/r$ in (4.1) becomes $\sin\theta \sin\psi$ and the volume element $\mathrm{d}^3\mathbf{r}$, becomes $r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\psi \, \mathrm{d}r$.

It is interesting to note that in (4.1) the correlation occurs between P in $\mathbf{x}, \mathbf{t}$ and in $\mathbf{x} + \mathbf{r}, t - r/c$. This correlation is essentially zero beyond a spatial separation l, the integral scale of the turbulence. The time needed for an acoustic signal to travel this distance, l/c, is so short with respect to the turbulent time l/u that we can consider the turbulent region as acoustically compact. Hence we may ignore in (4.1) the time shift r/c. We assume now, which is often done in turbulence studies, that the time and spatial dependence of P can be separated. If this is the case,

$$P(\mathbf{x}, t)\dot{P}(\mathbf{x} + \mathbf{r}, t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{P(\mathbf{x}, t)P(\mathbf{x} + \mathbf{r}, t)\} \tag{4.2}$$

$$P(\mathbf{x}, t)\dot{P}(\mathbf{x} + \mathbf{r}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\{P(\mathbf{x}, t)P(\mathbf{x} + \mathbf{r}, t)\} - \left\{\frac{\mathrm{d}}{\mathrm{d}t}P(\mathbf{x}, t)\frac{\mathrm{d}}{\mathrm{d}t}P(\mathbf{x} + \mathbf{r}, t)\right\}. \tag{4.3}$$

Upon time averaging (the overbar in (4.1)), only the last term on the right-hand side of (4.3) remains. Performing the time derivation, as in the previous section, by multiplying with u/l, and assuming a spatial correlation as $\exp(-r^2/l^2)$, which is a reasonable assumption for homogeneous turbulence (see *e.g.* Townsend [8]), we write

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}, t)\frac{\mathrm{d}}{\mathrm{d}t}P(\mathbf{x} + \mathbf{r}, t)} = \rho^2\left(\frac{u}{l}\right)^2 u^2 U^2 \exp(-r^2/l^2) \tag{4.4}$$

We introduce this into the second integral on the right-hand side of (4.1) and using the coordinate system described above, we obtain for the intensity I emitted through a unit surface at $\mathbf{x}$ in the direction $\mathbf{n}$,

$$I = nK\rho^2\left(\frac{u}{l}\right)^2 u^2 U^2 \int_a^\infty r^2\,\mathrm{d}r \int_0^\pi \mathrm{d}\psi \int_0^\pi \frac{\exp\{-(r/l)^2\}\sin^2\theta\sin\psi\,\mathrm{d}\theta}{rc}. \tag{4.5}$$

Using the definition (1.3) of α and (3.5) of K and taking the total radiation through a surface L^2, rather than a unit surface, we obtain after integration

$$L^2 I = \frac{1}{8}\alpha(\rho u U^2 L^2)\left(\frac{\rho U^2}{\gamma p_0}\right) M \left(\frac{u}{U}\right)^3 \exp(-a^2/l^2). \tag{4.6}$$

Whereas the emissions discussed so far, *i.e.* (2.8) found by CW and (3.15) found here, are proportional to L^3, this one is proportional to L^2. The reason for that is that the pressure correlation vanishes beyond distances of the order l. Therefore, the emission through the surface is proportional to $L^2 l$. Also, for this case we can formulate the gain G, that is (following [7]) the sum of the emitted radiation with and without bubbles, divided by the latter. From (1.1) and (4.6) we find, approximating $\exp(-a^2/l^2)$ by unity

$$G = 1 + \frac{1}{8}\alpha\left(\frac{\rho U^2}{\gamma p_0}\right) M^{-4}\left(\frac{U}{u}\right)^2 \frac{l}{L}. \tag{4.7}$$

Compared with the right-hand side of (3.17), giving the corresponding result for the correlation between pressure and velocity fluctuations coming from the same bubbles, the most important difference is the dependence on l/L. If we take values, pertaining to the experiments by [7] and assume for (U/u) the value 10, we obtain

$$G = 1 + 3 \times 10^9 \frac{l}{L}\alpha. \tag{4.8}$$

With $l = 1\,\text{cm}$ and $L = 1\,\text{km}$, the right-hand side of (4.8) is still comparable with that of (3.19). This calculation shows that the interaction between the original turbulence and the additional fluctuations brought about by the bubbles deserves further study. This should take into account the nonlinear effects which are (S. Lele, private communication) of importance near the boundary of the turbulent blob.

Dedication

This paper is dedicated to my friend and colleague Pieter Zandbergen, at the occasion of his 65th birthday. My congratulations are accompanied by wishing him many years of good health to come.

References

1. M. J. Lighthill, On sound generated aerodynamically, 1. General theory. *Proc. R. Soc. London* A211 (1952) 564–587.
2. D. G. Crighton and J. E. Ffowcs Williams, Sound generation by turbulent two-phase flow. *J. Fluid Mech.* 36 (1969) 585–603.
3. A. Prosperetti, Bubble-related ambient noise in the ocean. *J. Acoust. Soc. Am.* 84 (1988) 1042–1054.
4. D. G. Crighton, A. P. Dowling, J. E. Ffowcs Williams, M. Heckl, and F. G. Leppington, *Modern Methods in Analytical Acoustics.* Berlin: Springer (1992) 38pp.
5. G. K. Batchelor, Transport properties of two-phase materials with random structure. *Ann. Rev. Fluid Mech.* 6 (1974) 227–255.
6. L. Van Wijngaarden, One-dimensional flow of liquids containing small gas bubbles. *Ann. Rev. Fluid Mech.* 4 (1972) 369–396.
7. M. S. Korman, R. A. Roy and L. Crum, Enhancement of hydrodynamic noise radiation by the regulation of air bubbles in a turbulent water jet. *Proc. 14 Intern. Conf. On Acoust. Paper* B6-1 (1992) Beijing.
8. A. A. Townsend, *The Structure of Turbulent Shear Flow.* Cambridge: Cambridge University Press (1975) 29pp.